# RL with Linear Features: When Does It Work & When Doesn't It Work?

## Part 1: The Assumption Ladder & Bellman Completeness

CS 2284: Foundations of Reinforcement Learning

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# Agenda

**Announcements**

- Second reading assignment is out.

**Recap++**

- Wrap up our tabular sample complexity analysis (Chapter 2).
- **Minimax Result:** We established the fundamental limits of tabular learning.

**Today**

- Function Approximation. **We'll move beyond tabular RL!**

# Motivation: Beyond Tabular RL

**Recap: Tabular MDPs**

- State space $\mathcal{S}$, Action space $\mathcal{A}$.
- Sample complexity scales with $|\mathcal{S}||\mathcal{A}|$.
- **Problem:** In many real-world applications (robotics, games, healthcare), $|\mathcal{S}|$ is enormous or continuous.

**Function Approximation**

- Introduce a feature map $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$.
- Approximate values using linear functions:

$$f_\theta(s, a) = \theta^\top \phi(s, a)$$

- **Goal:** Sample complexity polynomial in $d$ (and $H$), independent of $|\mathcal{S}|$.

# Finite-horizon dynamic programming (reminder)

Stage-$h$ **optimal Bellman operator**:

$$(\mathcal{T}_h f)(s,a) \; := \; r_h(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)}\Big[\max_{a' \in \mathcal{A}} f(s',a')\Big].$$

Optimal $Q$ satisfies backward recursion:

$$Q^\star_H \equiv 0, \qquad Q^\star_h = \mathcal{T}_h Q^\star_{h+1} \quad \text{for } h = H-1, \dots, 0.$$

# Least-Squares Value Iteration (LSVI)

**Setting:** Finite Horizon $H$, Generative Model (Simulator).

**Algorithm:** Backward Induction via Regression.

1. Initialize $\widehat{V}_H(s) = 0$.
2. For $h = H - 1, \ldots, 0$:
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   - **Regression:** Solve for parameters $\widehat{\theta}_h$:

$$\widehat{\theta}_h \in \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \sum_{i=1}^N \left(\theta^\top \phi(s_i, a_i) - y_i\right)^2$$

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   - **Update:** Set $\widehat{Q}_h(s, a) = \widehat{\theta}_h^\top \phi(s, a)$ and $\widehat{V}_h(s) = \max_a \widehat{Q}_h(s, a)$.

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- **But RL is different:** LSVI is a **composition** of regressions.
  - The target for stage $h$ depends on our *own estimate* at $h+1$:

$$\text{Target}_h(s,a) \approx r(s,a) + \mathbb{E}\left[\max_{a'} \widehat{Q}_{h+1}(s',a')\right]$$

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  - **Crucial Question:** Even if $Q^\star$ is linear, does the target defined by $\widehat{Q}_{h+1}$ remain learnable (linear)?
- If the target "falls off the manifold", how do errors compound?

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(D) **Linear Bellman Completeness**
$\mathcal{T}_h f \in \mathcal{F}$ for all $f \in \mathcal{F}$.
*Status:* **Sufficient!** *(This Lecture)*

# Assumption (D): Linear Bellman Completeness

**Definition (Bellman Completeness)**
For any linear function $f(s,a) = w^\top \phi(s,a)$, applying the Bellman optimality operator $\mathcal{T}_h$ yields a function that is also linear in $\phi$.

$$\forall w \in \mathbb{R}^d, \exists \theta \in \mathbb{R}^d \text{ such that}$$

$$\underbrace{r_h(s,a) + \mathbb{E}_{s' \sim P_h(\cdot|s,a)}\left[\max_{a'} w^\top \phi(s',a')\right]}_{(\mathcal{T}_h f_w)(s,a)} = \theta^\top \phi(s,a)$$

**Key Implication:**

- If we run LSVI with finite samples, the **target** is always realizable.
- This reduces RL to a sequence of well-specified regression problems.

# Examples of Completeness

When does Linear Bellman Completeness hold?

1. **Tabular MDPs**
   $\phi(s, a)$ is a one-hot encoding of size $|\mathcal{S}||\mathcal{A}|$.
   Any function over $\mathcal{S} \times \mathcal{A}$ is linear in $\phi$.

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2. **Linear MDPs (Low-Rank Transition)**

$$P_h(s'|s, a) = \sum_{i=1}^{d} \phi_i(s, a)\mu_i(s')$$

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3. **Linear Quadratic Regulators (LQR)?**
   Yes, with quadratic features: $\phi(s, a) = [s^\top, a^\top, s^\top s, \dots]^\top$.
   (Value functions are quadratic in $s, a$).

*Warning: Adding "junk" features can break completeness!*

# The Error Propagation Question (Intuition)

Suppose we run approximate dynamic programming (like LSVI) where we force every estimate $\widehat{Q}_h$ to be a linear function:

$$\widehat{Q}_H \equiv 0, \qquad \widehat{Q}_h \leftarrow \text{Project}_{\mathcal{F}}\left(\mathcal{T}_h \widehat{Q}_{h+1}\right).$$

**The Subtle Danger:**

- Even if $Q^\star$ is linear, the **target** $\mathcal{T}_h \widehat{Q}_{h+1}$ might *not* be linear!
- If the target falls "off-manifold" (outside $\mathcal{F}$), we incur a projection error (bias) at step $h$.
- **The Crucial Question:** How do these errors compound?
  - Does the error at $h+1$ get amplified when we back up to $h$?
  - Without **Completeness** (closure), this error can grow exponentially with $H$.

*Realizability of $Q^\star$ alone does not guarantee the intermediate targets remain learnable.*

# Refresher: LSVI Regression Step

To analyze the algorithm, let's focus on exactly what happens at stage $h$.

We have a fixed next-stage value $\widehat{V}_{h+1}$. We gather data $D_h$ and solve:

$$\widehat{\theta}_h = \underset{\theta \in \mathbb{R}^d}{\text{argmin}} \sum_{i=1}^{N} \Bigg( \underbrace{\theta^\top \phi(s_i, a_i)}_{\text{Prediction}} - \underbrace{(r_i + \widehat{V}_{h+1}(s_i'))}_{\text{Target } y_i} \Bigg)^2$$

**Why does this work?**

- **Completeness** implies the *true expected target* is linear:

$$\mathbb{E}[y_i \mid s_i, a_i] = (\mathcal{T}_h \widehat{Q}_{h+1})(s_i, a_i) = \theta_h^\star \phi(s_i, a_i)$$

- So this is **well-specified** linear regression!
- Bias is zero; we only need to control variance.

# Fixed Design OLS (The Tool)

Consider the standard linear regression setting:

$$y_i = x_i^\top \theta^\star + \xi_i, \quad \text{with } \mathbb{E}[\xi_i | x_i] = 0 \text{ (sub-Gaussian).}$$

The OLS estimator is $\widehat{\theta} = \Lambda^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} x_i y_i \right)$, where $\Lambda = \frac{1}{N} \sum_i x_i x_i^\top$.

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## Fixed-design OLS Bound

With probability at least $1-\delta$, the prediction error is bounded in the $\Lambda$-norm:

$$\|\widehat{\theta} - \theta^\star\|_\Lambda \lesssim \sigma\sqrt{\frac{d\log(1/\delta)}{N}}.$$

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## Fixed-design OLS Bound

With probability at least $1 - \delta$, the prediction error is bounded in the $\Lambda$-norm:

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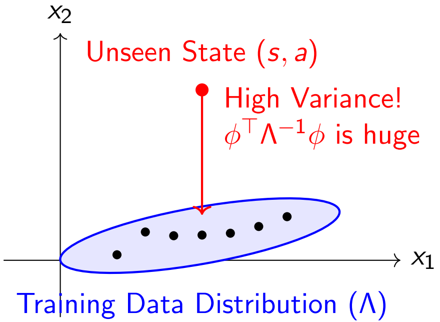
This translates to a *pointwise* bound using leverage scores:

$$|(\widehat{\theta} - \theta^\star)^\top \phi(s,a)| \leq \|\widehat{\theta} - \theta^\star\|_\Lambda \sqrt{\phi(s,a)^\top \Lambda^{-1} \phi(s,a)}$$

# The Hidden Failure Mode of OLS

**The Problem:**

- The OLS bound depends on $\Lambda = \frac{1}{N}\sum \phi(s_i, a_i)\phi(s_i, a_i)^\top$.
- It bounds the **average** prediction error (weighted by training data).
- RL requires **Uniform ($\ell_\infty$)** error bounds. We must predict well at *any* state the optimal policy might visit.




Backward induction may query *outside* the ellipse, causing huge expansion.

# D-optimal design: the leverage-minimizing geometry

To guarantee uniform bounds, we must choose our training data carefully.

The feature set is:

$$\Phi := \{\phi(s,a) : (s,a) \in \mathcal{S} \times \mathcal{A}\} \subset \mathbb{R}^d.$$

## D-optimal design (lemma; geometric fact)

Suppose $\Phi$ is compact. There exists a distribution $\rho$ supported on at most $d(d+1)/2$ state–action pairs s.t. with

$$\Sigma := \mathbb{E}_{(s,a)\sim\rho}\left[\phi(s,a)\phi(s,a)^\top\right],$$

we have $\Sigma \succ 0$ and

$$\sup_{(s,a)} \; \phi(s,a)^\top \Sigma^{-1} \phi(s,a) \;=\; d.$$

Furthermore, no distirbution $\rho$ can achive a lower (worst-case) leverage score.

# Geometric intuition

**Leverage control:** the quantity $\phi^\top \Sigma^{-1} \phi$ is exactly the (population) leverage

Equivalent viewpoints (pick your favorite story):

- **Kiefer–Wolfowitz:** $\rho$ maximizes $\log\det\left(\mathbb{E}_\rho[\phi\phi^\top]\right)$
- **John's ellipsoid:** the ellipsoid

$$\mathcal{E} = \{v : \; v^\top \Sigma^{-1} v \le d\}$$

is the minimum-volume centered ellipsoid containing $\Phi$

Message: there is always a way to sample from only $O(d^2)$ points while keeping worst-case leverage $\le d$.

# From Global to Pointwise Error

Sample $N$ points from the D-optimal design $\rho$. Then $\Lambda = \frac{1}{N}\sum \phi\phi^\top$, and our empirical cov is $\Lambda \approx \Sigma$.

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1. **The Leverage Score Bound (Geometry):** Since $\Lambda \approx \Sigma$, D-optimal design guarantees:

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2. **The OLS Bound (Statistics):** From the first slide, we know $\|\widehat{\theta} - \theta^\star\|_\Lambda \lesssim \sigma \sqrt{\frac{d}{N}}$.

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3. **Resulting Pointwise Guarantee:** Use $|\text{pointwise error}| \leq \|\widehat{\theta} - \theta^\star \cdot \|_\Lambda \sqrt{\text{Leverage}}$:

$$\sup_{(s,a)} |\widehat{Q}_h(s,a) - \mathcal{T}_h \widehat{Q}_{h+1}(s,a)| \;\lesssim\; \left(\sigma\sqrt{\frac{d}{N}}\right)\cdot\sqrt{d} \;=\; \frac{\sigma d}{\sqrt{N}}$$

This allows us to control the *max-norm* Bellman residual!

# Returning to LSVI

We now have all the pieces to analyze LSVI.

1. **The Data Collection (Generative Model)**
   - For each stage $h$, we don't just sample randomly.
   - We compute the D-optimal design $\rho^\star$ on $\Phi$.
   - We query the simulator $N$ times distributed according to $\rho^\star$.

2. **The Rough Sketch of the Proof**

$$\text{Regression Error} \xrightarrow{\text{D-Opt}} \text{Pointwise Error} \xrightarrow{\text{Completeness}}$$

$$\text{Bellman Residual} \xrightarrow{\text{Sim. Lemma}} \text{Policy Loss}$$

# The Main Theorem (Informal)

**Theorem (LSVI with Generative Model)**
Assume Linear Bellman Completeness. If we set:

$$N \approx \frac{H^6 d^2}{\epsilon^2}$$

and collect data using D-optimal design, then LSVI returns a policy $\widehat{\pi}$ such that with high probability:

$$V^\star(s_0) - V^{\widehat{\pi}}(s_0) \leq \epsilon$$

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**Takeaway:** We have achieved sample complexity polynomial in $d$ and $H$, independent of $|\mathcal{S}|$!

- **Completeness** ensures realizability.
- **D-Optimal Design** ensures uniform error control.

# Summary & Looking Ahead

**Today:** Scaling RL to large state spaces (using features)

- **The Algorithm:** Dynamic Programming as a sequence of regression problems (LSVI)
- **The Assumption Ladder:** consider different natural structural assumptions
- **Sampling:** use **D-Optimal Design** to control the uniform ($\ell_\infty$) error
- **Main Result: Linear BC + D-Optimal Design** is sufficient for $\mathrm{poly}(d, H)$ sample complexity.

**Next Time (Lecture 2):**

- **Rigorous Analysis:**
- **Offline RL:** adapt LSVI when we cannot choose our sampling distribution (Coverage Assumptions).