Policy Gradient: Optimality

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CS 2824: Foundations of Reinforcement Learning

Summary/Today

- Do they PG methods globally converge to an optimal policy?
  $\theta^{(t+1)} = \theta^{(t)} + \eta \nabla_\theta V^{(t)}(\mu)$

- Recap++
- Today:
  - Landscape result & Exploration
  - Tabular case
  - Natural policy gradient

Recap++

Things to remember

$$\nabla_\theta J(\theta) := \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a\,|\,s) Q^{\pi_\theta}(s,a)\right]$$

Today: we use $d^\pi_{s_0}$ for a state distribution measure.

(for convenience, we overload notation)

$$d^\pi_{s_0}(s) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s\,|\,s_0,\pi)$$

$$d^\pi_{s_0}(s,a) = (1-\gamma)\sum_{h=0}^{\infty}\gamma^h \mathbb{P}(s_h = s, a_h = a\,|\,s_0,\pi)$$

for a distribution over states $s \sim \mu$, let:

$$V^\pi(\mu) = E_{s\sim\mu}[V^\pi(s)]$$

$$d^\pi_\mu(s) = E_{s_0\sim\mu}[d^\pi_{s_0}(s)]$$

Derivation of unbiased Stochastic Policy Gradient

$$\nabla_\theta J(\theta) := \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a\,|\,s) Q^{\pi_\theta}(s,a)\right]$$

Draw $h \propto \gamma^h$, **roll-in** π_θ to generate $s_h, a_h \sim \mathbb{P}_h^{\pi_\theta}$

Roll-out π_θ from (s_h, a_h) : terminate with prob $1-\gamma$, $\widetilde{Q}^{\pi_\theta}(s_h, a_h) = \sum_{\tau=h}^{t\geq h} r_\tau$

Unbiased estimate: $\nabla_\theta \ln \pi_\theta(a_h\,|\,s_h)\,\widetilde{Q}^{\pi_\theta}(s_h, a_h)$

Policy Gradient: Examples of Policy Parameterization (discrete actions)

1. Softmax Policy for Tabular MDPs:

$$\theta_{s,a} \in \mathbb{R}, \forall s, a \in S \times A$$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta_{s,a})}{\sum_{a'} \exp(\theta_{s,a'})}$$

2. Softmax linear Policy (e.g., for linear MDPs):

Feature vector $\phi(s,a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s,a))}{\sum_{a'} \exp(\theta^\top \phi(s,a'))}$$

3. Neural Policy:

Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s,a))}{\sum_{a'} \exp(f_\theta(s,a'))}$$

PG as non-convex optimization

Convergence to Stationary Points of GD

$J(\pi_\theta)$ is non-convex

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- Def of a β-smooth function F:
  $\|\nabla_\theta F(\theta) - \nabla_\theta F(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$
  which implies:
  $$\left| F(\theta) - F(\theta_0) - \nabla_\theta F(\theta_0)^\top (\theta - \theta_0) \right| \leq \frac{\beta}{2} \|\theta - \theta_0\|_2^2$$

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- Proposition: (stationary point convergence) Assume $F(\theta)$ is β-smooth.
  Suppose we run gradient ascent: $\theta_{t+1} = \theta_t + \eta \nabla_\theta F(\theta_t)$, with $\eta = 1/(2\beta)$. Then:
  $$\min_{t \leq T} \|\nabla_\theta F(\theta_t)\|_2^2 \leq \frac{2\beta\left(\max_\theta F(\theta) - F(\theta_0) \right)}{T}$$

Convergence to Stationary Point

$J(\pi_\theta)$ is non-convex (see example in the monograph)

Def of β-smooth:

$$\|\nabla_\theta J(\theta) - \nabla_\theta J(\theta_0)\|_2 \leq \beta \|\theta - \theta_0\|_2$$

$$\left| J(\theta) - J(\theta_0) - \nabla_\theta J(\theta_0)^\top (\theta - \theta_0) \right| \leq \frac{\beta}{2} \|\theta - \theta_0\|_2^2, \forall \theta, \theta_0$$

[**Theorem**] If $J(\theta)$ is β-smooth, and we run SGA: $\theta_{t+1} = \theta_t + \eta \widetilde{\nabla}_\theta J(\theta_t)$

where $\mathbb{E}\left[\widetilde{\nabla}_\theta J(\theta_t)\right] = \nabla_\theta J(\theta_t), \quad \mathbb{E}\left[\|\widetilde{\nabla}_\theta J(\theta_t)\|_2^2\right] \leq \sigma^2,$

then:

$$\mathbb{E}\left[\frac{1}{T}\sum_t \|\nabla_\theta J(\theta_t)\|_2^2\right] \leq O\left(\sqrt{\beta\sigma^2/T}\right)$$

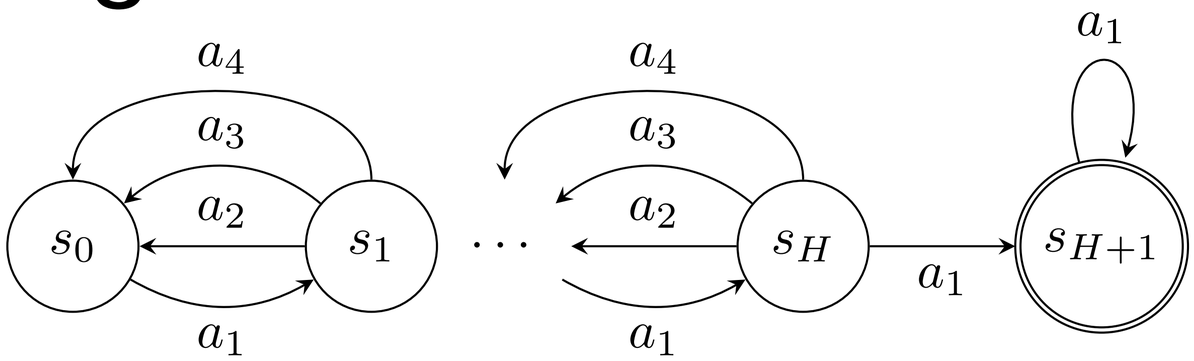
Today:

When do PG methods converge to a global optima?
(+ what about function approximation?)

A “landscape” result

(and “exploration”)

Vanishing Gradients and Saddle Points



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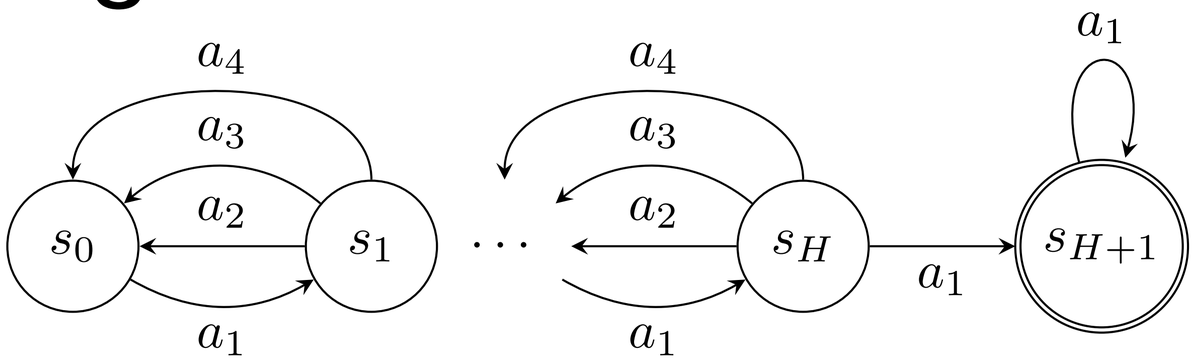

Set $\gamma = H/(H+1)$. Policy param:

for $a = a_1, a_2, a_3$, $\pi_\theta(a \mid s) = \theta_{s,a}$, and $\pi_\theta(a_4 \mid s) = 1 - \theta_{s,a_1} - \theta_{s,a_2} - \theta_{s,a_3}$

(this a “direct” param, which is valid inside the simplex)

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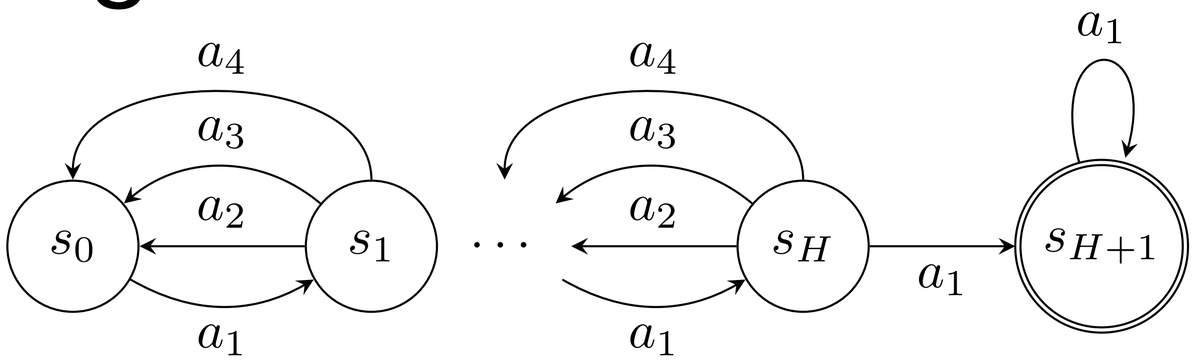

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Theorem: For $0 < \theta < 1$ (componentwise) and $\theta_{s,a_1} < 1/4$ (for all states s).

For all $k \leq O(H/\log(H))$, we have that

$$\|\nabla_\theta^k V^{\pi_\theta}(s_0)\| \leq (1/3)^{H/4}$$

(where $\|\nabla_\theta^k V^{\pi_\theta}(s_0)\|$ is the operator norm of the tensor $\nabla_\theta^k V^{\pi_\theta}(s_0)$.

“Vanilla” PG for the Softmax

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- Given our a starting distribution ρ over states, recall our objective is:
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- While we are interested in good performance under ρ, it is helpful to optimize under a different measure μ. Specifically, consider optimizing: $V^{\pi_\theta}(\mu)$, i.e.
  $\max_{\theta \in \Theta} V^{\pi_\theta}(\mu)$,
  even though our ultimate goal is performance under $V^{\pi_\theta}(\rho)$.

The Softmax Policy Class

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- We have that:
  $$\frac{\partial \log \pi_\theta(a \mid s)}{\partial \theta_{s',a'}} = \mathbf{1}\left[s = s'\right]\left(\mathbf{1}\left[a = a'\right] - \pi_\theta(a' \mid s)\right)$$
  where $\mathbf{1}[\,\cdot\,]$ is the indicator function.

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- Lemma: For the softmax policy class, we have:
  $$\frac{\partial V^{\pi_\theta}(\mu)}{\partial \theta_{s,a}} = \frac{1}{1-\gamma} d_\mu^{\pi_\theta}(s) \pi_\theta(a \mid s) A^{\pi_\theta}(s,a)$$

Proof

$$
\begin{aligned}
\frac{\partial V^{\pi_\theta}(\mu)}{\partial \theta_{s,a}} &= E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \nabla_\theta \ln \pi_\theta(a \mid s) A^{\pi_\theta}(s,a) \right] \\
&= E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \mathbf{1}[s_t = s] \Big(\mathbf{1}[a_t = a] A^{\pi_\theta}(s,a) - \pi_\theta(a \mid s) A^{\pi_\theta}(s_t, a_t) \Big) \right] \\
&= E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\sum_{t=0}^{\infty} \gamma^t \mathbf{1}[(s_t, a_t) = (s,a)] A^{\pi_\theta}(s,a) \right] + \pi_\theta(a \mid s) \sum_{t=0}^{\infty} \gamma^t E_{\tau \sim \Pr_\mu^{\pi_\theta}} \left[\mathbf{1}[s_t = s] A^{\pi_\theta}(s_t, a_t) \right] \\
&= \frac{1}{1-\gamma} E_{(s',a') \sim d^{\pi_\theta}} \left[\mathbf{1}[(s',a') = (s,a)] A^{\pi_\theta}(s,a) \right] + 0 \\
&= \frac{1}{1-\gamma} d^{\pi_\theta}(s,a) A^{\pi_\theta}(s,a) \,,
\end{aligned}
$$

Aside: The Performance Difference Lemma

For all π, π', s_0:

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma} \mathbb{E}_{s \sim d^{\pi}_{s_0}} \mathbb{E}_{a \sim \pi(\cdot|s)} \left[A^{\pi'}(s, a) \right]$$

$$d^{\pi}_{s_0}(s) = (1-\gamma) \sum_{h=0}^{\infty} \gamma^h \mathbb{P}(s_h = s \,|\, s_0, \pi)$$

Proof: The Performance Difference Lemma

$$
\begin{aligned}
V^{\pi}(s) - V^{\pi'}(s) &= \mathbb{E}_{\tau \sim \Pr^{\pi}(\cdot|s_0=s)} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \right] - V^{\pi'}(s) \\
&= \mathbb{E}_{\tau \sim \Pr^{\pi}(\cdot|s_0=s)} \left[\sum_{t=0}^{\infty} \gamma^t \left(r(s_t, a_t) + \gamma V^{\pi'}(s_{t+1}) - V^{\pi'}(s_t) \right) \right] \\
&= \mathbb{E}_{\tau \sim \Pr^{\pi}(\cdot|s_0=s)} \left[\sum_{t=0}^{\infty} \gamma^t \left(r(s_t, a_t) + \gamma \mathbb{E}[V^{\pi'}(s_{t+1}) \mid s_t, a_t] - V^{\pi'}(s_t) \right) \right] \\
&= \mathbb{E}_{\tau \sim \Pr^{\pi}(\cdot|s_0=s)} \left[\sum_{t=0}^{\infty} \gamma^t \left(Q^{\pi'}(s_t, a_t) - V^{\pi'}(s_t) \right) \right] \\
&= \mathbb{E}_{\tau \sim \Pr^{\pi}(\cdot|s_0=s)} \left[\sum_{t=0}^{\infty} \gamma^t A^{\pi'}(s_t, a_t) \right].
\end{aligned}
$$

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- Theorem: Assume the μ is strictly positive i.e. $\mu(s) > 0$ for all states s. For $\eta \leq (1-\gamma)^3/8$, then we have that for all states $s, V^{(t)}(s) \to V^\star(s), \text{ as } t \to \infty$.

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- Comments:
  - rate could be exponentially slow in S, H.
  - need $\mu > 0$ is necessary.

PG+Log Barrier Regularization

(for the softmax)

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- Consider the log barrier λ-regularized objective:
$$L_\lambda(\theta) := V^{\pi_\theta}(\mu) - \lambda\, E_{s\sim \mathrm{Unif}_S}\left[\mathrm{KL}(\mathrm{Unif}_A, \pi_\theta(\,\cdot\,|\,s))\right]$$
$$= V^{\pi_\theta}(\mu) + \frac{\lambda}{SA}\sum_{s,a}\log \pi_\theta(a\,|\,s) + \lambda \log A$$

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- Do small gradients imply a globally optimal policy?

Stationarity and Optimality

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Stationarity and Optimality

- Log barrier regularized objective:

$$L_\lambda(\theta) = V^{\pi_\theta}(\mu) + \frac{\lambda}{SA}\sum_{s,a}\log \pi_\theta(a\,|\,s) + \lambda \log A$$

- Theorem: (Log barrier regularization) Suppose θ is such that:

$\|\nabla_\theta L_\lambda(\theta)\|_2 \leq \epsilon_{opt}$ and $\epsilon_{opt} \leq \lambda/(2SA)$

then we have for all starting state distributions ρ:

$$V^{\pi_\theta}(\rho) \geq V^\star(\rho) - \frac{2\lambda}{1-\gamma}\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty$$

where the "distribution mismatch coefficient" is

$$\left\|\frac{d_\rho^{\pi^\star}}{\mu}\right\|_\infty = \max_s\left(\frac{d_\rho^{\pi^\star}(s)}{\mu(s)}\right)$$ (componentwise division notation)

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- Corollary: (Iteration complexity with log barrier regularization)
  Set $\lambda = \dfrac{\epsilon(1-\gamma)}{2\left\| \frac{d_\rho^{\pi^\star}}{\mu} \right\|_\infty}$ and $\eta = 1/\beta_\lambda$. Starting from any initial $\theta^{(0)}$,
  then for all starting state distributions ρ, we have

  $$\min_{t<T} \left\{ V^\star(\rho) - V^{(t)}(\rho) \right\} \leq \epsilon \quad \text{whenever} \quad T \geq c \frac{S^2 A^2}{(1-\gamma)^6 \, \epsilon^2} \left\| \frac{d_\rho^{\pi^\star}}{\mu} \right\|_\infty^2$$

  (for constant c).

Proof, part 1 (optional)

- The proof consists of showing that: $\max_a A^{\pi_\theta}(s,a) \leq 2\lambda/(\mu(s)S)$ for all states s.

- To see that this is sufficient, observe that by the performance difference lemma:

$$
\begin{aligned}
V^\star(\rho) - V^{\pi_\theta}(\rho) &= \frac{1}{1-\gamma} \sum_{s,a} d_\rho^{\pi^\star}(s)\pi^\star(a \mid s) A^{\pi_\theta}(s,a) \\
&\leq \frac{1}{1-\gamma} \sum_{s} d_\rho^{\pi^\star}(s) \max_{a \in A} A^{\pi_\theta}(s,a) \\
&\leq \frac{1}{1-\gamma} \sum_{s} 2 d_\rho^{\pi^\star}(s) \lambda/(\mu(s)S) \\
&\leq \frac{2\lambda}{1-\gamma} \max_s \left(\frac{d_\rho^{\pi^\star}(s)}{\mu(s)} \right).
\end{aligned}
$$

which would then complete the proof.

Proof, part 2 (optional)

- need to show $A^{\pi_\theta}(s,a) \le 2\lambda/(\mu(s)S)$ for all (s,a). consider (s,a) where that $A^{\pi_\theta}(s,a) \ge 0$ (else claim is true).
- Recall $\dfrac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} = \dfrac{1}{1-\gamma} d_\mu^{\pi_\theta}(s)\pi_\theta(a\,|\,s)A^{\pi_\theta}(s,a) + \dfrac{\lambda}{S}\left(\dfrac{1}{A} - \pi_\theta(a\,|\,s)\right)$
- Solving for $A^{\pi_\theta}(s,a)$ in the first step and using $\|\nabla_\theta L_\lambda(\theta)\|_2 \le \epsilon_{opt} \le \lambda/(2SA)$,

$$A^{\pi_\theta}(s,a) = \frac{1-\gamma}{d_\mu^{\pi_\theta}(s)}\left(\frac{1}{\pi_\theta(a\,|\,s)}\frac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} + \frac{\lambda}{S}\left(1 - \frac{1}{\pi_\theta(a\,|\,s)A}\right)\right)$$

$$\le \frac{1-\gamma}{d_\mu^{\pi_\theta}(s)}\left(\frac{1}{\pi_\theta(a\,|\,s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right)$$

$$\le \frac{1}{\mu(s)}\left(\frac{1}{\pi_\theta(a\,|\,s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right) \qquad \text{using that} \quad d_\mu^{\pi_\theta}(s) \ge (1-\gamma)\mu(s)$$

- Suppose we could show that $\pi_\theta(a\,|\,s) \ge 1/(2A)$, when $A^{\pi_\theta}(s,a) \ge 0$, then

$$\frac{1}{\mu(s)}\left(\frac{1}{\pi_\theta(a\,|\,s)}\frac{\lambda}{2SA} + \frac{\lambda}{S}\right) \le \frac{1}{\mu(s)}\left(2A\frac{\lambda}{2SA} + \frac{\lambda}{S}\right) = \frac{2\lambda}{\mu(s)S}$$

and the proof is done!

Proof, part 3 (optional)

- for (s, a) such that $A^{\pi_\theta}(s, a) \geq 0$, we want show $\pi_\theta(a \mid s) \geq 1/(2A)$.

- The gradient norm assumption $\|\nabla_\theta L_\lambda(\theta)\|_2 \leq \epsilon_{opt}$ implies that:

$$\epsilon_{opt} \geq \frac{\partial L_\lambda(\theta)}{\partial \theta_{s,a}} = \frac{1}{1-\gamma} d_\mu^{\pi_\theta}(s) \pi_\theta(a \mid s) A^{\pi_\theta}(s, a) + \frac{\lambda}{S}\left(\frac{1}{A} - \pi_\theta(a \mid s)\right)$$

$$\geq 0 + \frac{\lambda}{S}\left(\frac{1}{A} - \pi_\theta(a \mid s)\right) \qquad \text{using} \quad A^{\pi_\theta}(s, a) \geq 0$$

- Rearranging and using our assumption $\epsilon_{opt} \leq \lambda/(2SA)$,

$$\pi_\theta(a \mid s) \geq \frac{1}{A} - \frac{\epsilon_{opt} S}{\lambda} \geq \frac{1}{2A}\,.$$

The Natural Policy Gradient

- Recall that the Fisher information matrix of a parameterized density $p_\theta(x)$ is defined as $E_{x\sim p_\theta}\left[\nabla \log p_\theta(x) \nabla \log p_\theta(x)^\top\right]$

The Natural Policy Gradient

- Recall that the Fisher information matrix of a parameterized density $p_\theta(x)$ is defined as $E_{x\sim p_\theta}\left[\nabla \log p_\theta(x) \nabla \log p_\theta(x)^\top\right]$
- Define $\mathscr{F}_\rho^\theta$ as the (average) Fisher matrix on the family of distributions $\{\pi_\theta(\,\cdot\,|\,s)\,|\,s \in S\}$ as: $\mathscr{F}_\rho^\theta := E_{s\sim d_\rho^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[(\nabla \log \pi_\theta(a\,|\,s))\nabla \log \pi_\theta(a\,|\,s)^\top\right]$.

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- The NPG algorithm performs gradient updates in this induced geometry: $\theta^{(t+1)} = \theta^{(t)} + \eta F_\rho(\theta^{(t)})^\dagger \nabla_\theta V^{(t)}(\rho)$,

  where $M^\dagger$ denotes the Moore-Penrose pseudoinverse of M.

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- Idea:
  - ‘stretch’ the corners of the simplex out to travel faster
    (as opposed to the log-barrier which keeps us away)

“Compatible Function Approximation” (and NPG)

Compatible Function Approximation

- Let $w^\star$ denote the following minimizer of the "compatible function approximation" error:

$$w^\star \in \text{argmin}_w E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[\left(A^{\pi_\theta}(s,a) - w\cdot\nabla_\theta \log \pi_\theta(a\,|\,s)\right)^2\right]$$

Compatible Function Approximation

- Let $w^\star$ denote the following minimizer of the "compatible function approximation" error:

$$w^\star \in \text{argmin}_w E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[\left(A^{\pi_\theta}(s,a) - w\cdot\nabla_\theta \log \pi_\theta(a\,|\,s)\right)^2\right]$$

- Lemma: Let $\widehat{A}^{\pi_\theta}(s,a)$ be the best linear predictor of $A^{\pi_\theta}(s,a)$ using $\nabla_\theta \log \pi_\theta(a\,|\,s)$, i.e. $\widehat{A}^{\pi_\theta}(s,a) := w^\star \cdot \nabla_\theta \log \pi_\theta(a\,|\,s)$. We have:

$$\nabla_\theta V^{\pi_\theta}(\mu) = \frac{1}{1-\gamma} E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[\nabla_\theta \log \pi_\theta(a\,|\,s)\, \widehat{A}^{\pi_\theta}(s,a)\right]$$

We can use $\widehat{A}^{\pi_\theta}(s,a)$ instead of $A^{\pi_\theta}(s,a)$.

Proof

- The first order optimality conditions for $w^\star$ imply

$$E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[\left(A^{\pi_\theta}(s,a) - w^\star \cdot \nabla_\theta \log \pi_\theta(a\,|\,s)\right) \nabla_\theta \log \pi_\theta(a\,|\,s)\right] = 0$$

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- Rearranging and using the definition of $\widehat{A}^{\pi_\theta}(s,a)$,

$$\begin{aligned}
\nabla_\theta V^{\pi_\theta}(\mu) &= \frac{1}{1-\gamma} E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[A^{\pi_\theta}(s,a)\, \nabla_\theta \log \pi_\theta(a\,|\,s)\right] \\
&= \frac{1}{1-\gamma} E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)}\left[\left(w^\star \cdot \nabla_\theta \log \pi_\theta(a\,|\,s)\right) \nabla_\theta \log \pi_\theta(a\,|\,s)\right] \\
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NPG & Compatible Function Approximation

- Let $w^\star$ denote the following minimizer of the "compatible function approximation" error:

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- Lemma: We have that $F_\mu(\theta)^\dagger \nabla_\theta V^\theta(\mu) = \frac{1}{1-\gamma} w^\star,$

  The NPG direction is the weights $w^\star$

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- By the definition of $\nabla_\theta V^\theta(\mu)$ and $F_\mu(\theta)$:

$$(1-\gamma)\,\nabla_\theta V^\theta(\mu) = F_\mu(\theta) w^\star$$

Softmax Case:
NPG and Global Convergence to Opt

NPG softmax case

(NPG as “soft” policy iteration)

- Lemma: (Softmax NPG as soft policy iteration) The NPG update is:

$$\theta^{(t+1)} = \theta^{(t)} + \frac{\eta}{1-\gamma} A^{(t)}$$

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- Lemma: (Softmax NPG as soft policy iteration) The NPG update is:

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- and this leads to the update:

$$\pi^{(t+1)}(a \mid s) = \pi^{(t)}(a \mid s) \frac{\exp\left(\eta A^{(t)}(s,a)/(1-\gamma)\right)}{Z_t(s)},$$

where $Z_t(s) = \sum_a \pi^{(t)}(a \mid s) \exp\left(\eta A^{(t)}(s,a)/(1-\gamma)\right)$.

Proof

- Recall that:

$$F_\mu(\theta)^\dagger \nabla_\theta V^\theta(\mu) = \frac{1}{1-\gamma} w^\star$$

where

$$w^\star \in \text{argmin}_w E_{s\sim d_\mu^{\pi_\theta}} E_{a\sim\pi_\theta(\cdot|s)} \left[\left(A^{\pi_\theta}(s,a) - w \cdot \nabla_\theta \log \pi_\theta(a \mid s) \right)^2 \right]$$

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- Also, recall that:

$$\frac{\partial \log \pi_\theta(a\,|\,s)}{\partial \theta_{s',a'}} = \mathbf{1}\left[s = s'\right]\left(\mathbf{1}\left[a = a'\right] - \pi_\theta(a'\,|\,s)\right)$$

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- What is a minimizer for this square loss problem?

Global convergence for NPG

- Theorem: Params: $\theta^{(0)} = 0$ and $\eta > 0$. For all ρ and $T > 0$, we have:

$$V^{(T)}(\rho) \geq V^{\star}(\rho) - \frac{\log A}{\eta T} - \frac{1}{(1-\gamma)^2 T}.$$

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  - No ‘flat gradient’ problem
- What about approx/estimation errors? (next lecture)

Improvement Lower Bound

- Lemma: For the iterates $\pi^{(t)}$ generated by the NPG, we have for all distributions μ:

$$V^{(t+1)}(\mu) - V^{(t)}(\mu) \geq \frac{(1-\gamma)}{\eta} E_{s\sim\mu} \log Z_t(s) \geq 0.$$

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- Proof: First, let us show that $\log Z_t(s) \geq 0$. To see this, observe:

$$\begin{aligned}
\log Z_t(s) &= \log \sum_a \pi^{(t)}(a \,|\, s) \exp(\eta A^{(t)}(s,a)/(1-\gamma)) \\
&\geq \sum_a \pi^{(t)}(a \,|\, s) \log \exp(\eta A^{(t)}(s,a)/(1-\gamma)) \\
&= \frac{\eta}{1-\gamma} \sum_a \pi^{(t)}(a \,|\, s) A^{(t)}(s,a) = 0.
\end{aligned}$$

(using Jensen's inequality on the concave function $\log x$.)

Lemma Proof: continued….

By the performance difference lemma,

$$
\begin{aligned}
V^{(t+1)}(\mu) - V^{(t)}(\mu) &= \frac{1}{1-\gamma} E_{s\sim d_\mu^{(t+1)}} \sum_a \pi^{(t+1)}(a \mid s) A^{(t)}(s,a) \\
&= \frac{1}{\eta} E_{s\sim d_\mu^{(t+1)}} \sum_a \pi^{(t+1)}(a \mid s) \log \frac{\pi^{(t+1)}(a \mid s) Z_t(s)}{\pi^{(t)}(a \mid s)} \\
&= \frac{1}{\eta} E_{s\sim d_\mu^{(t+1)}} \mathrm{KL}(\pi_s^{(t+1)} \,||\, \pi_s^{(t)}) + \frac{1}{\eta} E_{s\sim d_\mu^{(t+1)}} \log Z_t(s) \\
&\geq \frac{1}{\eta} E_{s\sim d_\mu^{(t+1)}} \log Z_t(s) \geq \frac{1-\gamma}{\eta} E_{s\sim \mu} \log Z_t(s),
\end{aligned}
$$

where the last step uses that $d_\mu^{(t+1)} \geq (1-\gamma)\mu$ and that $\log Z_t(s) \geq 0$.

NPG Conv. Proof, Part 1

- $d^\star$ as shorthand for $d_\rho^\star$; π_s as shorthand for the vector of $\pi(\,\cdot\,|\,s)$

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- $d^\star$ as shorthand for $d_\rho^\star$; π_s as shorthand for the vector of $\pi(\cdot \mid s)$
- By the performance difference lemma,

$$V^{\pi^\star}(\rho) - V^{(t)}(\rho) = \frac{1}{1-\gamma} E_{s\sim d^\star} \sum_a \pi^\star(a \mid s) A^{(t)}(s,a)$$

$$= \frac{1}{\eta} E_{s\sim d^\star} \sum_a \pi^\star(a \mid s) \log \frac{\pi^{(t+1)}(a \mid s) Z_t(s)}{\pi^{(t)}(a \mid s)}$$

$$= \frac{1}{\eta} E_{s\sim d^\star} \left(\mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t)}) - \mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t+1)}) + \sum_a \pi^*(a \mid s) \log Z_t(s) \right)$$

$$= \frac{1}{\eta} E_{s\sim d^\star} \left(\mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t)}) - \mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t+1)}) + \log Z_t(s) \right),$$

NPG Conv. Proof, Part 2

- By the improvement lemma $V^{(t+1)}(\rho) \geq V^{(t)}(\rho)$. Hence,

$$V^{\pi^\star}(\rho) - V^{(T-1)}(\rho) \leq \frac{1}{T}\sum_{t=0}^{T-1}(V^{\pi^\star}(\rho) - V^{(t)}(\rho))$$

$$= \frac{1}{\eta T}\sum_{t=0}^{T-1} E_{s\sim d^\star}(\mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t)}) - \mathsf{KL}(\pi_s^\star \,||\, \pi_s^{(t+1)})) + \frac{1}{\eta T}\sum_{t=0}^{T-1} E_{s\sim d^\star} \log Z_t(s)$$

$$\leq \frac{E_{s\sim d^\star}\mathsf{KL}(\pi_s^\star \,||\, \pi^{(0)})}{\eta T} + \frac{1}{\eta T}\sum_{t=0}^{T-1} E_{s\sim d^\star} \log Z_t(s)\,.$$

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- By the improvement lemma (applied with $d^\star$ as the distribution), we have:

$$\frac{1}{\eta} E_{s\sim d^\star}\log Z_t(s) \leq \frac{1}{1-\gamma}\left(V^{(t+1)}(d^\star) - V^{(t)}(d^\star)\right)$$

which gives us a bound on $E_{s\sim d^\star}\log Z_t(s)$.

NPG Conv. Proof, Part 3

$$
\begin{aligned}
V^{\pi^\star}(\rho) - V^{(T-1)}(\rho) &\leq \frac{E_{s\sim d^\star}\mathsf{KL}(\pi_s^\star \,||\, \pi^{(0)})}{\eta T} + \frac{1}{\eta T}\sum_{t=0}^{T-1} E_{s\sim d^\star} \log Z_t(s) \\
&\leq \frac{E_{s\sim d^\star}\mathsf{KL}(\pi_s^\star \,||\, \pi^{(0)})}{\eta T} + \frac{1}{(1-\gamma)T}\sum_{t=0}^{T-1} \left(V^{(t+1)}(d^\star) - V^{(t)}(d^\star) \right) \\
&= \frac{E_{s\sim d^\star}\mathsf{KL}(\pi_s^\star \,||\, \pi^{(0)})}{\eta T} + \frac{V^{(T)}(d^\star) - V^{(0)}(d^\star)}{(1-\gamma)T} \\
&\leq \frac{\log A}{\eta T} + \frac{1}{(1-\gamma)^2 T}\,.
\end{aligned}
$$