Natural Policy Gradient

Sham Kakade and Kianté Brantley

CS 6789: Foundations of Reinforcement Learning

Today:

Natural policy optimization

History:

A Natural Policy Gradient

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NeurIPS 2002

Covariant Policy Search

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IJCAI 2003

Trust Region Policy Optimization

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ICML 2015

Notations and Settings:

Finite horizon setting: $\mathscr{M} = \{S, A, H, r, P, \rho\}$

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Trajectory distribution:

$$\Pr^{\pi}(\tau) = \rho(s_0)\pi(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi(a_1 \mid s_1)\ldots P(s_{H-1} \mid s_{H-2}, a_{H-2})\pi(a_{H-1} \mid s_{H-1})$$

Revisit Gradient Descent:

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Different re-parameterization (scaling & translation) can lead to a quite different GD path

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$\theta = A\phi + b$ — ϕ – coordinates

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Policy Optimization:

$$\max_{\pi_\theta} V^{\pi_\theta}(\rho)$$

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Q: How to do second-order Taylor expansion on the KL constraint?

Let’s do second order Taylor Expansion on the KL-divergence

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$$\frac{1}{H} KL\left(\Pr^{\pi_{\theta_0}} || \Pr^{\pi_\theta}\right) = \frac{1}{H} \sum_\tau \Pr^{\theta_0}(\tau) \ln \frac{\Pr^{\theta_0}(\tau)}{\Pr^{\theta}(\tau)} = \frac{1}{H} \sum_\tau \Pr^{\theta_0}(\tau) \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_0}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)}$$

KL - divergence by definition

$$\frac{P_r \pi_{\theta_0}(\tau)}{P_r \pi_\theta(\tau)} = \frac{\mu(s_0)\, \pi_{\theta_0}(a_0|s_0)\, P(s_1|s_0,a_0)\, \pi_{\theta_0}(a_1|s_1) \cdots}{\mu(s_0)\, \pi_\theta(a_0|s_0)\, P(s_1|s_0,a_0)\, \pi_\theta(a_1|s_1) \cdots}$$

$$\ln \frac{P_r \pi_{\theta_0}(\tau)}{P_r \pi_\theta(\tau)} = \sum_{h=0}^{H-1} \ln \frac{\pi_{\theta_0}(a_h|s_h)}{\pi_\theta(a_h|s_h)}$$

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$$= \mathbb{E}_{s_h, a_h \sim d^{\pi_{\theta_0}}} \left[\ln \frac{\pi_{\theta_0}(a_h \,|\, s_h)}{\pi_{\theta}(a_h \,|\, s_h)} \right] := \ell(\theta)$$

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$$\nabla_\theta \ell(\theta)\big|_{\theta=\theta_0} = \mathbb{E}_{s \sim d^{\pi_{\theta_0}}} \sum_a \pi_{\theta_0}(a \mid s) \left(-\nabla_\theta \ln \pi_\theta(a_h \mid s_h)\big|_{\theta=\theta_0} \right) = 0$$

$$\ln \frac{\pi_{\theta_0}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)} = \ln \pi_{\theta_0}(a_h \mid s_h) - \ln \pi_\theta(a_h \mid s_h)$$

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$$= -\mathbb{E}_{s \sim d^{\pi_{\theta_0}}} \sum_a \pi_{\theta_0}(a \,|\, s) \frac{\nabla_\theta \pi_{\theta_0}(a \,|\, s)}{\pi_{\theta_0}(a \,|\, s)} = -\mathbb{E}_{s \sim d^{\pi_{\theta_0}}} \sum_a \nabla_\theta \pi_{\theta_0}(a|s)$$

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$\ln \pi_{\theta_0} - \ln \pi_\theta$

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$$\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{f\cdot g'}{g^2}$$

$$\nabla^2_\theta \ell(\theta)\,|_{\theta=\theta_0} = \mathbb{E}_{s\sim d^{\pi_{\theta_0}}} \sum_a \pi_{\theta_0}(a\,|\,s)\Big(-\nabla^2_\theta \ln \pi_\theta(a\,|\,s)\,|_{\theta=\theta_0}\Big)$$

$$\nabla_\theta \ln \pi_\theta(a|s) = \frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)}$$

$$\nabla_\theta\left[\nabla_\theta \ln \pi_\theta(a|s)\right] = \nabla_\theta\left[\frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)}\right]$$

$$= \frac{\nabla^2_\theta \pi(a|s)}{\pi(a|s)} - \frac{\nabla_\theta \pi(a|s)\,\nabla_\theta \pi(a|s)^T}{\pi^2_\theta(a|s)}$$

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$= 0$

$$\sum_a \pi_{\theta_0}(a|s) \cdot \frac{\nabla^2_\theta \pi_{\theta_0}(a|s)}{\pi_{\theta_0}(a|s)}$$

$$\nabla^2_\theta \pi_{\theta_0}(a|s) = \nabla_\theta\left[\nabla_\theta \pi_{\theta_0}(a|s)\right] = \nabla_\theta\Big[\nabla_\theta \underbrace{\textstyle\sum_a \pi(a|s)}_{1}\Big]$$

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$= 0$

$$\frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)} \cdot \frac{\nabla_\theta \pi_\theta(a|s)^\top}{\pi_\theta(a|s)}$$

$$= \mathbb{E}_{s,a\sim d^{\pi_{\theta_0}}}\Big[\nabla_\theta \ln \pi_{\theta_0}(a\,|\,s)\Big(\nabla_\theta \ln \pi_{\theta_0}(a\,|\,s)\Big)^\top\Big]$$

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Fisher Information Matrix!

Second-order Taylor Expansion of KL at θ_0

$$\frac{1}{H} KL\left(\mathrm{Pr}^{\pi_{\theta_0}} || \mathrm{Pr}^{\pi_\theta}\right) \leq \delta \Rightarrow \frac{1}{2}(\theta - \theta_0)^\top F_{\theta_0}(\theta - \theta_0) \leq \delta$$

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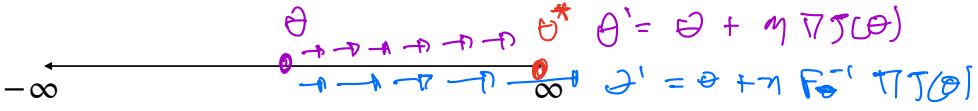

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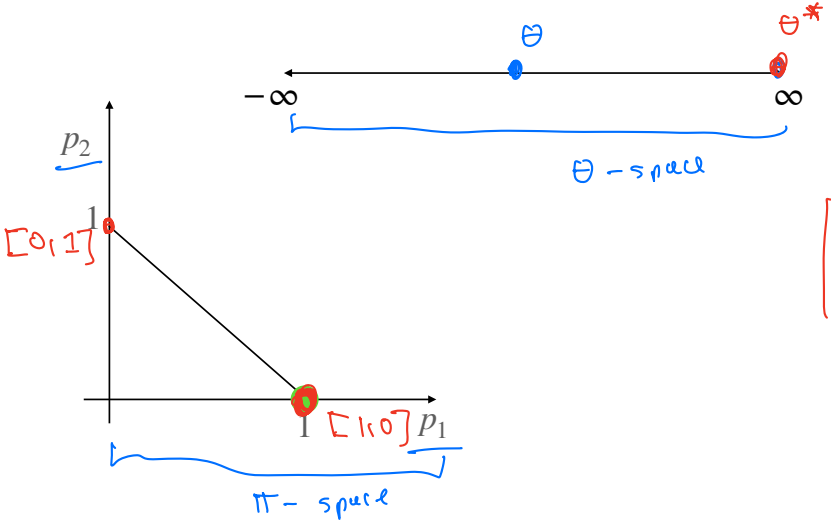

$$p_\theta = \left(\frac{\exp(\theta)}{1+\exp(\theta)}, \frac{1}{1+\exp(\theta)}\right)$$

$$\pi_\theta = \frac{\exp(\theta)}{\sum \exp(\theta)}$$

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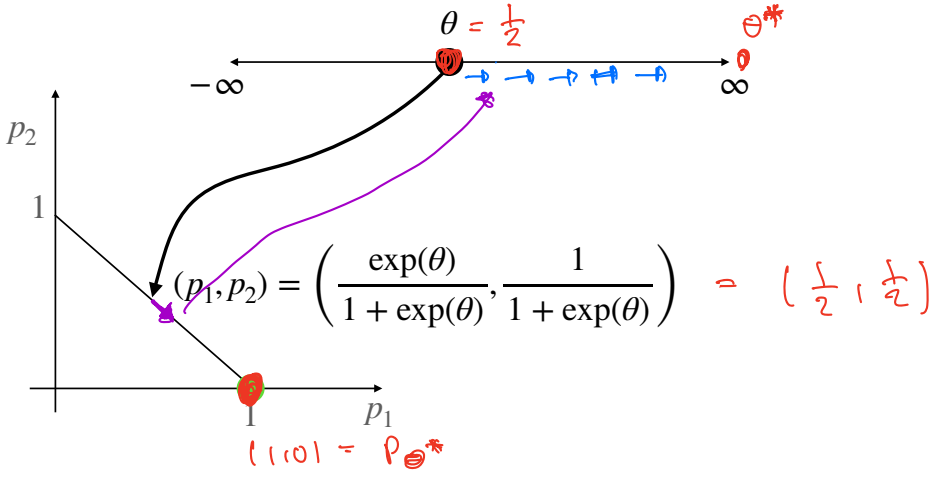

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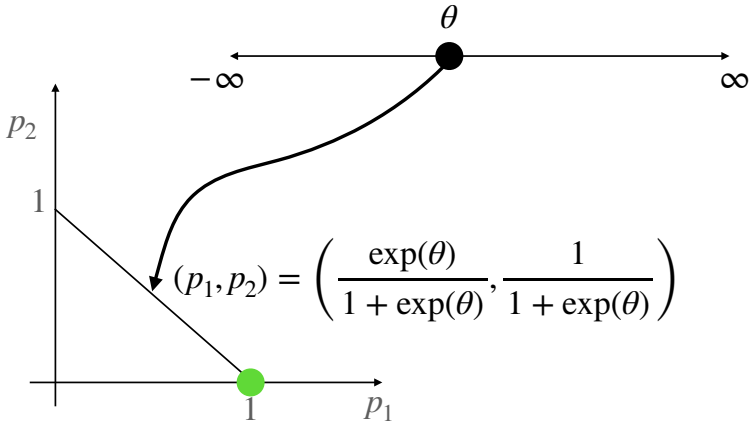
fisher information scalar

$$f_\theta = \frac{\exp(\theta_0)}{(1+\exp(\theta_0))^2}$$

$F_\theta \to 0^+$, as $\theta \to \infty$



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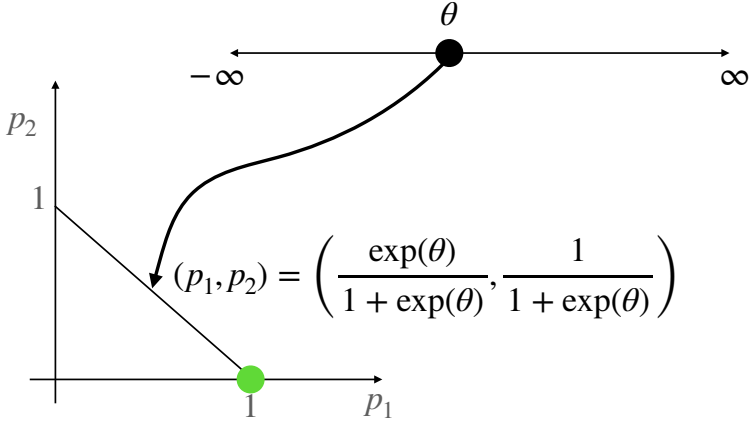

$$F_\theta \to 0^+, \text{as } \theta \to \infty$$

$$F_{\theta_0}(\theta - \theta_0)^2 \leq \delta \Rightarrow (\theta - \theta_0)^2 \leq \frac{\delta}{F_{\theta_0}} \to \infty, \text{as } \theta_0 \to \infty$$

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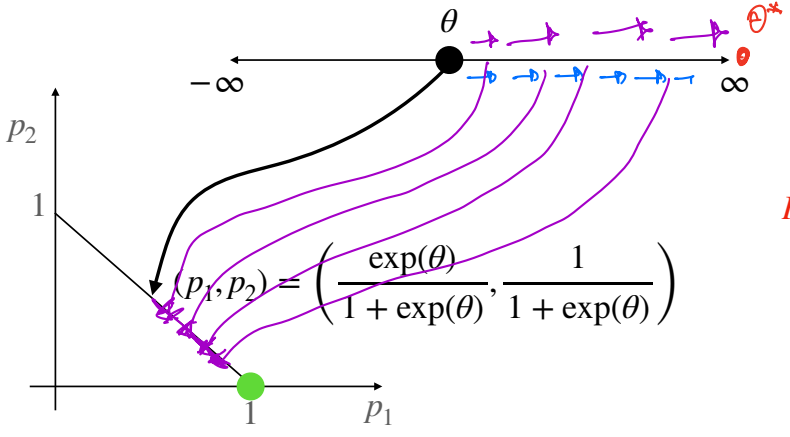

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Plain GD in θ will move to $\theta = \infty$ at a constant speed, while Natural GD can traverse faster and faster when θ gets bigger (Infinitely fast when $\theta \to \infty$)

Now we can solve the following quadratic programming:

PG

$$\max_{\theta} \nabla V^{\pi_{\theta_0}}(\rho)^{\top}\left(\theta - \theta_0\right)$$

$$\text{s.t.} \left(\theta - \theta_0\right)^{\top} F_{\theta_0}(\theta - \theta_0) \leq \delta$$

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We have a closed form solution:

$$\theta = \theta_0 + \sqrt{\frac{\delta}{(\nabla V^{\pi_{\theta_0}})^{\top} F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}}} \cdot F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}$$

Now we can solve the following quadratic programming:

$$\max_{\theta} \nabla V^{\pi_{\theta_0}}(\rho)^{\top}\left(\theta - \theta_0\right)$$

$$\text{s.t. } \left(\theta - \theta_0\right)^{\top} F_{\theta_0}(\theta - \theta_0) \leq \delta$$

We have a closed form solution:

$$\theta = \theta_0 + \sqrt{\frac{\delta}{(\nabla V^{\pi_{\theta_0}})^{\top} F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}}} \cdot F_{\theta_0}^{-1} \nabla V^{\pi_{\theta_0}}$$

NPG

Self-normalized step-size
(Learning rate is adaptive)

Summary

Natural Policy Gradient invariant to linear transformation
(Trust region constraint in terms KL on trajectory distributions)

Second order Taylor expansion of $\ell(\theta) := KL\left(\text{Pr}^{\pi_{\theta_0}} || \text{Pr}^{\pi_\theta}\right)$ at θ_0 is $(\theta - \theta_0)^\top F_{\theta_0}(\theta - \theta_0)$