Multi-armed Bandits

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CS 2824: Foundations of Reinforcement Learning

Announcements

Course Projects

- Check the course website for the project deadline dates

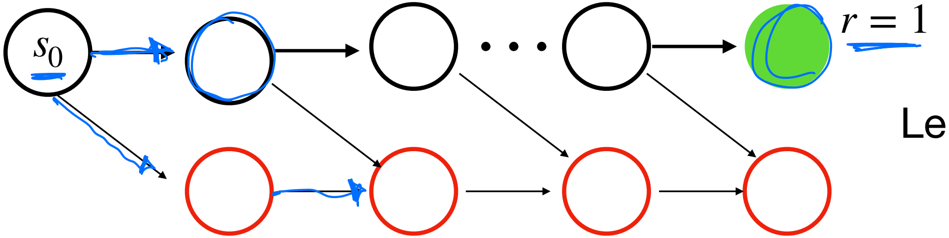
- Three important dates:
  - Proposal due after spring break
  - Midterm report due before project presentations
  - Final report due after reading period

- Groups
  - Group sizes are at ~~max 3~~ min 3
  - Group presentations will be in the last three lectures; we know this overlaps with ICLR, and we'll poll for presentation slots before ICLR.

The need for Exploration in RL:

The Combination Lock Example (i.e., the sparse reward problem)

(1) We have reward zero everywhere except at the goal (the right end);
(2) Every black node, one of the two actions will lead the agent to the dead state (red)



Length: H

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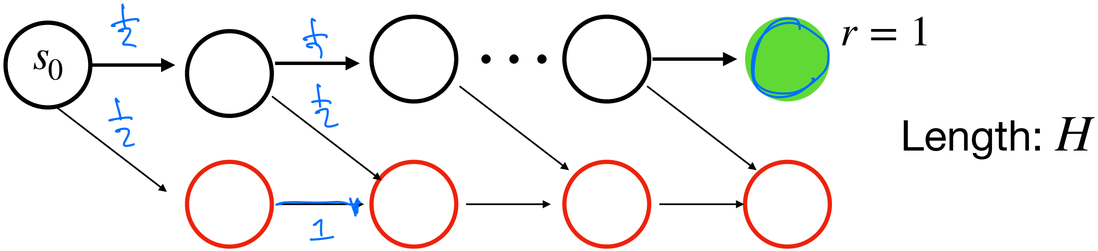


What is the probability of a random policy generating a trajectory that hits the goal?

$\left(\frac{1}{2}\right)^H = 2^{-H}$

Exploration!

We need to perform systematic exploration,
i.e., remember where we visited, and purposely try to visit unexplored regions..

What we will do today:

Study Exploration in a very simple MDP:

$$\mathscr{M} = \{s_0, \{a_1, \ldots, a_K\}, H = 1, R\}$$

i.e., MDP with one state, one-step transition, and K actions

This is also called Multi-armed Bandits

Plan for today:

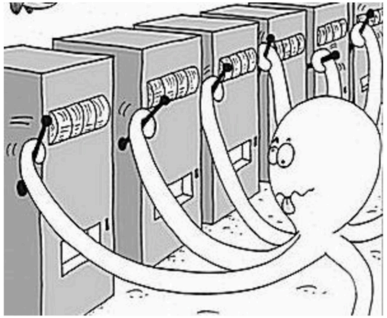
1. Introduction of MAB

2. Attempt 1: Greedy Algorithm (a bad algorithm)

3. Attempt 2: Explore and Commit

4. Attempt 3: Upper Confidence Bound (UCB) Algorithm

Intro to MAB



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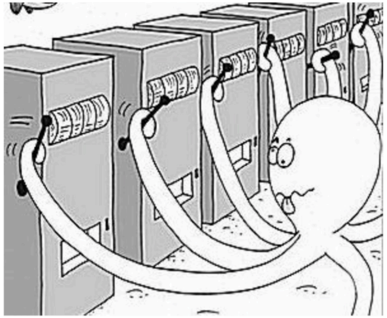

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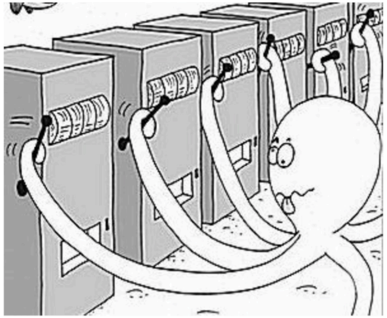

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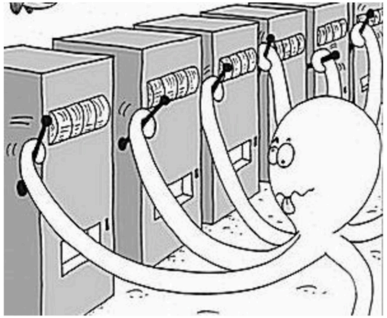

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Every time we pull arm a_i, we observe an i.i.d reward $r = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1-p \end{cases}$

Intro to MAB

Applications on online advertisement:

Arms correspond to Ads

Each arm has **click-through-rate** (CTR): probability of getting clicked (unknown)

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A learning system aims to maximize CTR in a long run:

1. **Try** an Ad (pull an arm)
2. **Observe** if it is clicked (see a zero-one **reward**)
3. **Update**: Decide what ad to recommend for next round

Intro to MAB

More formally, we have the following interactive learning process:

For $t = 0 \rightarrow T - 1$

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Note: each iteration, we do not observe rewards of arms that we did not try

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$$\text{Regret}_T = T\mu^\star - \sum_{t=0}^{T-1} \mu_{I_t}$$

$$\mu^\star = \max_{i \in [K]} \mu_i$$

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Goal: no-regret, i.e., $\text{Regret}_T / T \to 0$, as $T \to \infty$

Intro to MAB

Why the problem is hard?

Exploration and Exploitation Tradeoff:

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Exploration and Exploitation Tradeoff:

Every round, we need to ask ourselves:

Should we pull arms that are less frequently tried in the past (i.e., **explore**),
Or should we commit to the current best arm (i.e., **exploit**)?

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Alg: try each arm once, and then commit to the one that has the **highest observed** reward

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Q: what could be wrong?

A bad arm (i.e., low μ_i) may generate a high reward by chance!
(recall we have $r \sim \nu$, i.i.d)

Attempt 1: Greedy Algorithm

More concretely, let's say we have two arms a_1, a_2:

Reward dist for a_1: w/ prob 60%, $r = 1$; else $r = 0$ $\mu_1 = .6$

Reward dist for a_2: w/ prob 40%, $r = 1$; else $r = 0$ $\mu_2 = .4$

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$.4 \times .4 = 16\%$

But try a_1, a_2 once, with probability 16%, we will observe reward pair $(0,1)$

$(\mu^* - \mu_2)(T-2)$

$(.6 - .4)(T-2)$

$R_T = .2(T-2)$

$\frac{R_T}{T} = \frac{.2(T-2)}{T}$ $T \to \infty$ $\frac{R_T}{T} \to .2$

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The greedy alg will pick a_2—**loosing expected reward 0.2 every time in the future**

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(a bad algorithm: constant regret)

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Due to randomness in the reward distribution, trying each arm once is not enough, i.e., observed single reward may be far away from the mean

Q: What's the fix here?

Yes, let's (1) try each arm multiple times, (2) compute the empirical mean of each arm, (3) commit to the one that has the highest empirical mean

Q: How many times is enough?

Alg: Explore and Commit:

times that we pull each arm k

Algorithm hyper parameter $N < T/K$ (we assume $T >> K$)

For $k = 1 \to K$: (# Exploration phase)

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$N = \frac{T}{K}$

$NK = \frac{T}{K} \cdot K = T$

For $t = NK \rightarrow T - 1$: (# Exploitation phase)

Pull the best empirical arm, i.e., $I_t = \arg\max_{i \in [K]} \hat{\mu}_i$

Q: how to set N?

Statistical Tools:

1. Hoeffding inequality (optional, no need to remember or understand it)

$r \in [0, 1] \qquad r_1, \ldots, r_n \sim V.$

$$| \hat{\mu} - \mu | \leq \sqrt{\frac{1}{n}}$$

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Given a distribution $\nu \in \Delta([0,1])$, and N i.i.d samples $\{r_i\}_{i=1}^N \sim \nu$, w/ probability at least $1-\delta$, we have:

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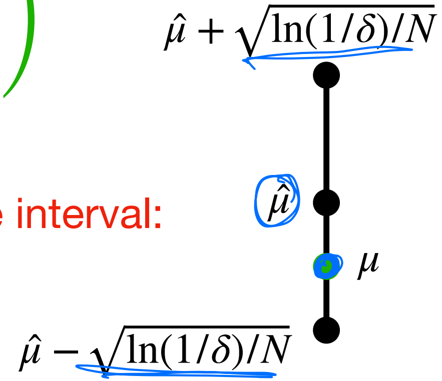

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After the Exploration phase, with probability at least 1-δ,

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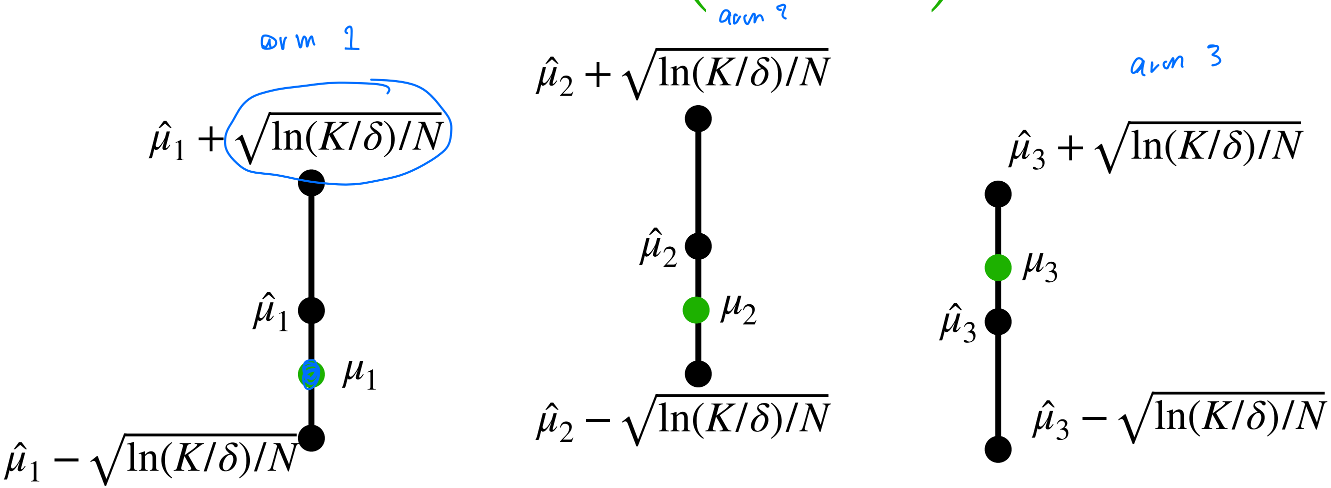

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Total regret

Regret$_{explore}$ + Regret$_{exploitation}$

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Let's now bound $\text{Regret}_{exploit}$

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$$\leq 2\sqrt{\ln(K/\delta)/N}$$

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$\hat{\mu}_{\hat{I}} \geq \hat{\mu}_{I^*} \Rightarrow 0 \geq \hat{\mu}_{\hat{I}} - \hat{\mu}_{I^*}$

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≤ 0

Q: why?

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T

Finally, combine two regret together:

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f(n) =

$$\text{Regret}_T = \text{Regret}_{explore} + \text{Regret}_{exploit} \leq NK + 2T\sqrt{\frac{\ln(K/\delta)}{N}}$$

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Minimize the upper bound via optimizing N:

Set $N = \left(\frac{T\sqrt{\ln(K/\delta)}}{2K}\right)^{2/3}$, we have:

$$\text{Regret}_T \leq O\left(T^{2/3}K^{1/3} \cdot \ln^{1/3}(K/\delta)\right)$$

$R_T = T^{2/3} \cdot K^{1/3}$

$\frac{R_T}{T} = \frac{T^{2/3} \cdot K^{1/3}}{T} = T^{-1/3}$

$T \to \infty$

$R_T \to 0$

To conclude on Explore then Commit:

[Theorem] Fix $\delta \in (0,1)$, set $N = \left(\frac{T\sqrt{\ln(K/\delta)}}{2K}\right)^{2/3}$, with probability at least $1-\delta$, **Explore and Commit** has the following regret:

$$\text{Regret}_T \leq O\left(T^{2/3}K^{1/3} \cdot \ln^{1/3}(K/\delta)\right)$$

Q: can we do better, particularly, can we get $\sqrt{T}$ regret bound?

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Recall the Tool for Building Confidence Interval:

We can show that for all iterations t, we have the for all $k \in [K]$, w/ prob $1 - \delta$,

$$|\hat{\mu}_k(i) - \mu_k| \leq \sqrt{\frac{\ln(KT/\delta)}{N_t(k)}}$$

empirical mean

Ground truth

N

UCB: Optimism in the face of Uncertainty

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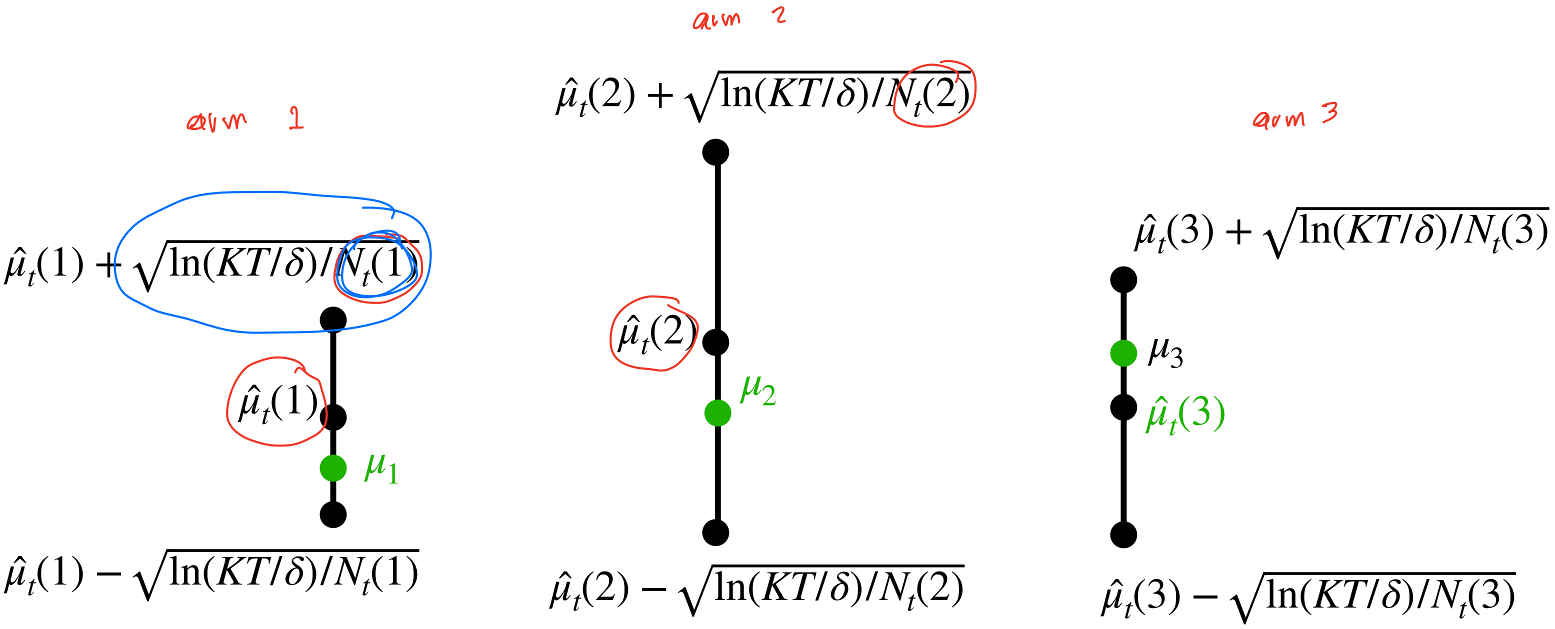

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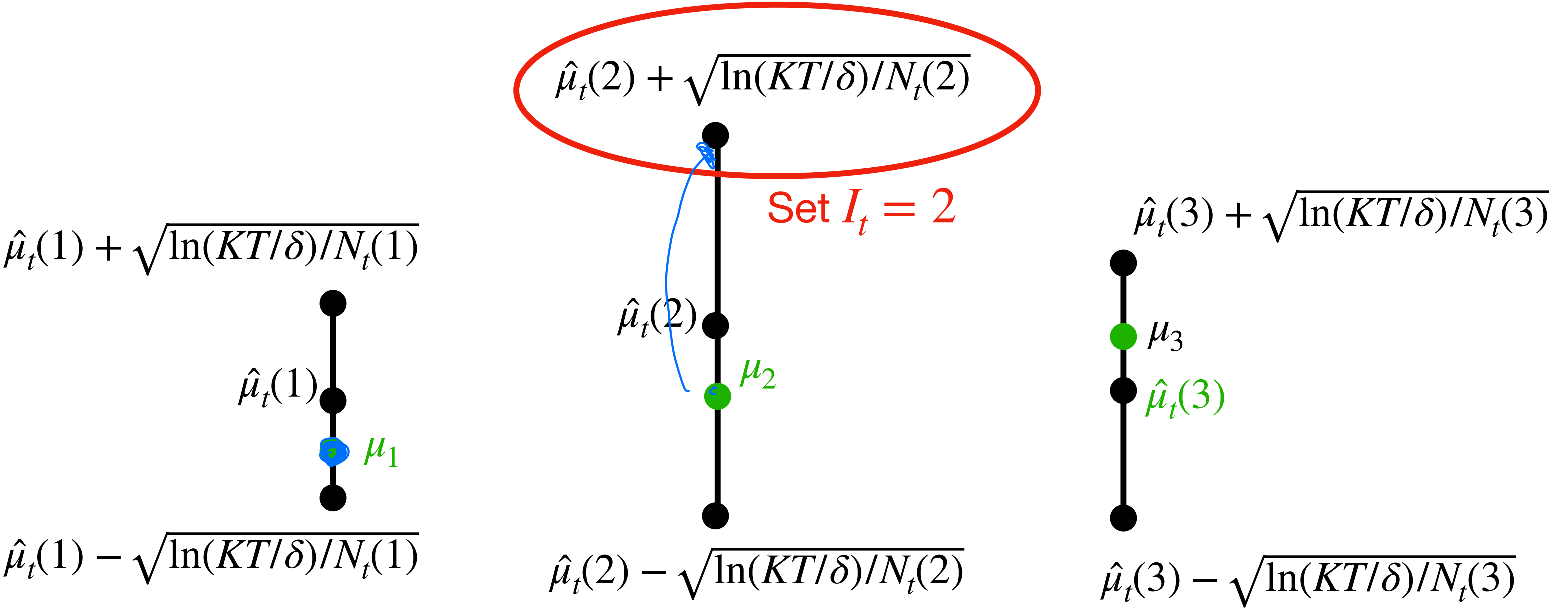

Put things together: UCB Algorithm:

For $t = 0 \to T - 1$:

$$I_t = \arg\max_{i \in [K]} \left(\hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}} \right)$$

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"Reward Bonus": $\sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

$N_t(i) \downarrow$ Bonus $\uparrow$

$N_t(i) \uparrow$ Bonus $\downarrow$

UCB Regret:

[Theorem (informal)] With high probability, UCB has the following regret:

$$\text{Regret}_T = \widetilde{O}\left(\sqrt{KT}\right)$$

Intuitive Explanation of UCB

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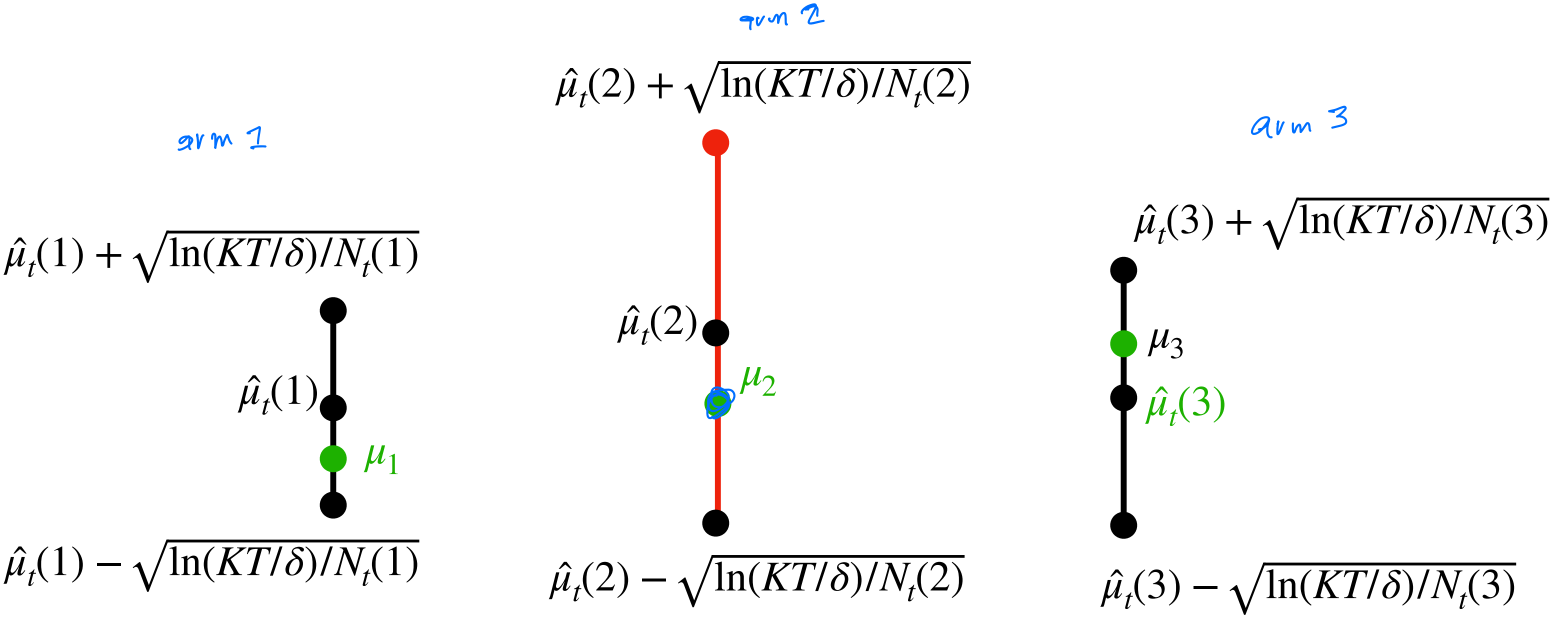
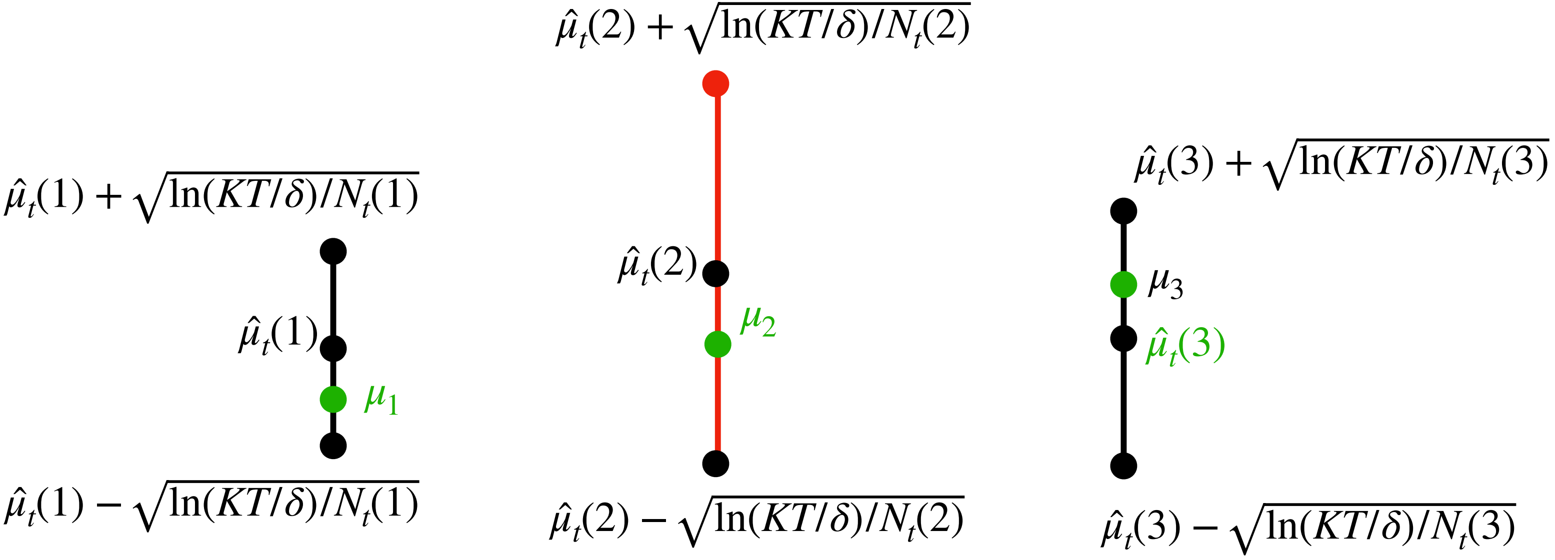
Q: why upper confidence is high for arm 2?



Intuitive Explanation of UCB

Case 1: it has large conf-interval, which means that it has not been tried many times yet (high uncertainty)



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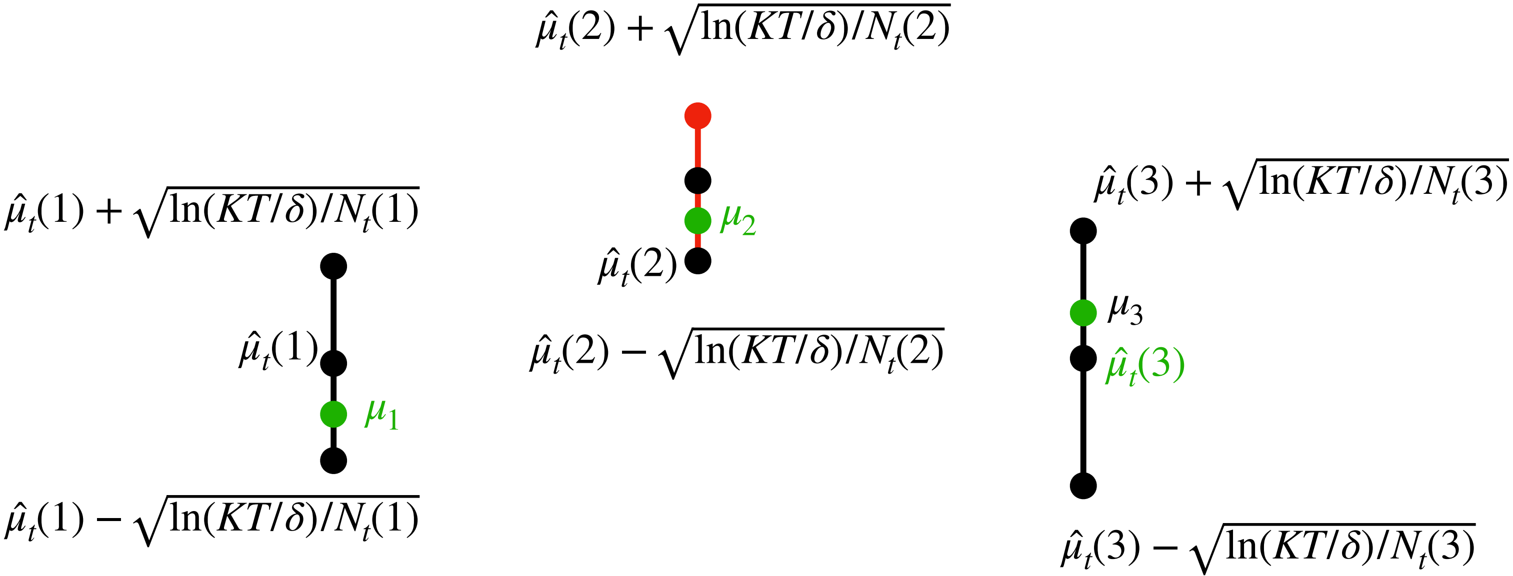
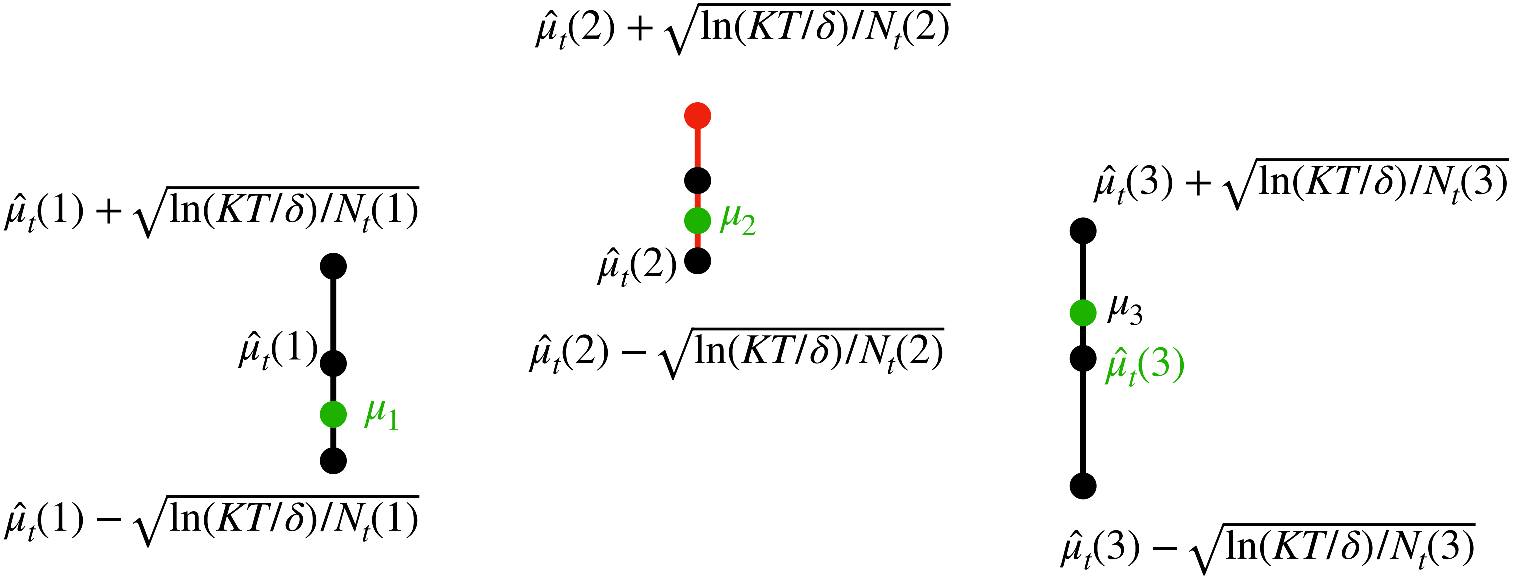


Intuitive Explanation of UCB

Case 2: it has low uncertainty, then it is simply a good arm, i.e., it's true mean is high!



Explore and Exploration Tradeoff

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Case 2: I_t has small conf-interval, then it is simply a good arm, i.e., it's true mean is pretty high!

Thus, we do exploitation in this case!

Let's formalize the intuition

Denote the optimal arm $I^\star = \arg\max_{i \in [K]} \mu_i$; recall $I_t = \arg\max_{i \in [K]} \hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

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Regret-at-t $= \mu^\star - \mu_{I_t}$